May 1
11 AM

Dear Bill

Here - on schedule - is my next epistle. As you may realize by now I mean business and I am putting things in their proper place! I have some committments (which I made in my more naive past) which must be completed - from then on I hope I will never be over such a barrel again. Enough of that guff - on with the war. The attachments are

Problem F - a discussion on the power of the velocity in the ~~of~~ drag term - as you may see, you sit about in the middle of the region $\sim v^{1.4}$ Actual data is your answer here!

Problem G Matchbox problem. I have now produced a better result than I gave you over the phone. I am quite pleased with the way it turned out and would certainly like to see how it checks out.

Concerning time - I am now far more than 3 days into the next 4 day period but we will count it as 3 days and will consider this extra time as part of my contribution to a joint paper or report if we write one. I found the problem interesting - that is the main thing.

Regards
Ev

P.S. The red ink only means I couldn't find a black ball point pen!

Problem F What is the power of the velocity which occurs in the drag term of the basic differential equation

$$m\frac{d^2x}{dt^2} + C\left(\frac{dx}{dt}\right)^n + kx = 0$$

References: The following material is taken primarily from Chemical Engineers Handbook (Fourth Edition) by Perry, Chilton and Kirkpatrick (5-59, Particle Dynamics section) - Mc Graw Hill publication.
The figure is taken from the third edition, page 1018

Drag force on a particle is given by

$$F_d = \frac{C A_p \rho u^2}{2 g_c}$$

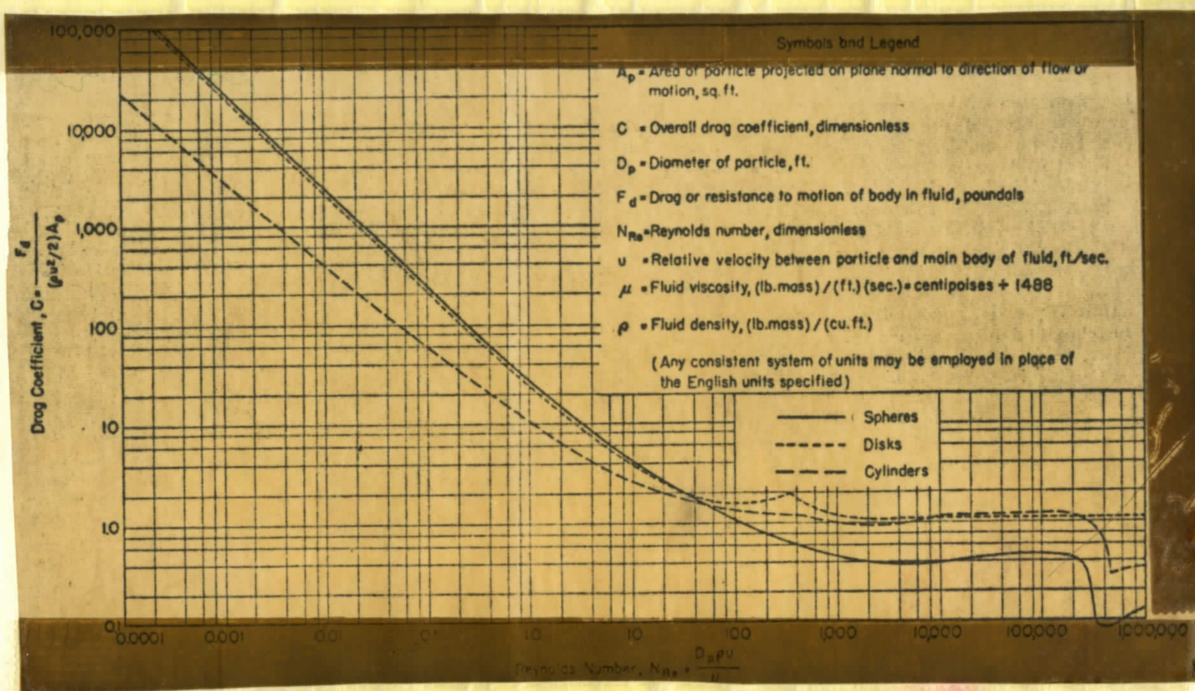
F_d ~ drag force in pounds
C ~ drag coefficient (dimensionless)
A_p ~ projected area of particle in direction of motion (square feet)
ρ ~ density of surrounding fluid in lb/cu ft.
u ~ relative velocity between particle and fluid ft/sec
$g = 32.17$ (lb)(ft)/(lb force)(sec)^2

Now - on the surface this looks as though the drag force varies ~~with~~ as u^2, however the catch is that C is a rather peculiar function! On the next page is a graph of C, for various shaped bodies, expressed as a function of the Reynolds number N_{Re}, where

$$N_{Re} = \frac{D_p \rho u}{\mu}$$

where D_p = diameter of particle (in feet)
ρ & u are as above
μ = Fluid viscosity in lb mass/ft sec

Note - μ can also be expressed in centipoises and then divided by 1488



Now we note that for $N_{Re} <$ ~~1.0~~ 0.3 we have, very closely $C = \frac{24}{N_{Re}}$

∴ In this region

$$F_d = \frac{\cancel{24}^{12}\,\mu}{D_p\cancel{\rho}\cancel{u}}\,\frac{A_p\cancel{\rho}\,u^{\cancel{2}}}{\cancel{2}g} = \frac{12}{D_p}\left(\frac{\pi}{4}D_p^2\right)\mu u$$

$$= \frac{3\pi\mu u D_p}{g} \quad \Big\}\ \text{Stokes Law}$$

$$\boxed{\text{Thus in this region, we have} \quad m\frac{d^2x}{dt^2} + \left(\frac{3\pi\mu D_p}{g}\right)\dot{x} + kx = 0 \quad \Big\}\ \text{or } n = 1\,!}$$

letting $u = \dot{x}$

Now this only holds for small particles or large particles moving at low velocities — it is especially useful & accurate in engineering practice in dust collection (ie air sent through centrifugal devices & dust particles ejected.)

For the region $0.3 < N_{Re} < 1000$ we find that

$$C = \frac{18.5}{N_{Re}^{0.6}}$$ which yields a

drag term of the type

$$\text{Constant}\,(\dot{x})^{1.4}$$

while for $1000 < N_{Re} < 200{,}000$ we have ~~to~~ a good approximation

$C \approx 0.44$ for spheres and here the drag term really is a $(\dot{x})^2$ term – I believe that most of your experiments lie in the $(\dot{x})^{1.4}$ region. (* See bottom of page)

This is tough to handle mathematically. You should gather the experimental data and when I get there we can look it over and see if it is worthwhile to look into the problem or at least have the computer boys look into it!

* For instance I recall that you mentioned the following conditions – body of several square ft cross section (say $D = 1\frac{1}{2}'$) – in air – Amplitude of oscillation $= 3/8''$ & $T = 5$ sec.

Here $\rho = (62.4)(0.0013)$ lbs/cu ft.

μ at room temp $\approx .0185$ centipoise $/ 1488$

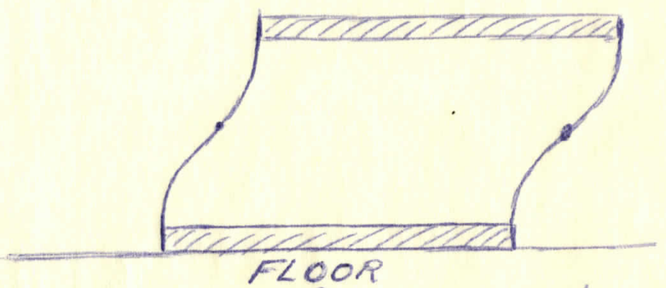
average $\dot{x} = \frac{2\pi r}{T} = \frac{2\pi}{5}\left(\frac{3}{8 \times 12}\right) \approx \frac{1}{25}$ ft/sec

$$\therefore N_{Re} \approx \left(\frac{3}{\cancel{2}}\right)(60 \times .0013)\left(\frac{1}{\cancel{25}}\right)\left(\frac{\overset{30}{\cancel{1500}}}{.0185}\right) = \frac{5400}{15} \approx 350$$

So even if the law is still applicable – you are now ~~outside the~~ between the $\dot{x}$ & $(\dot{x})^2$ regions. "Better you should ask Mother Nature for the answer" take it from me – "she knows!"

Problem G   More accurate solution of matchbox problem

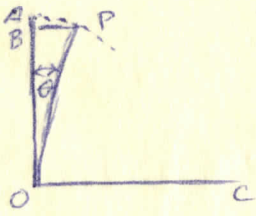
The sides, because of the manner in which they are attached to the heavy ends, actually assume a sigmoidal shape. As will be seen, the primary motion of the top plate is a horizontal one with only a slight vertical displacement.



In order to set up Lagranges equations we need an expression for the vertical motion of the mass at the top. We proceed as follows

1st approximation

Consider ~~the~~ a flexible side OA to be moved to a position OP (where we assume OP is virtually a straight line). Then the top of the flexible side drops a vertical distance AB where B is the foot of the perpendicular from P to OA.



$$AB = AO - BO = AO - OP\cos\theta$$
$$= AO(1-\cos\theta)$$
$$= AO\left(1-\frac{BO}{OP}\right)$$

Now if we let OC be the y axis and OA the x axis and P has the coordinates (y_0, x_0) then $BO = x_0$ and ~~OP = l (length of~~

$$AB = AO\left(1-\frac{x_0}{\sqrt{x_0^2+y_0^2}}\right) = AO\left(1-\frac{1}{\sqrt{1+\left(\frac{y_0}{x_0}\right)^2}}\right)$$

$$= AO\left(1-\left(1-\tfrac{1}{2}\left(\tfrac{y_0}{x_0}\right)^2 \cdots\right)\right) \approx \frac{AO}{2}\frac{y_0^2}{x_0^2}$$

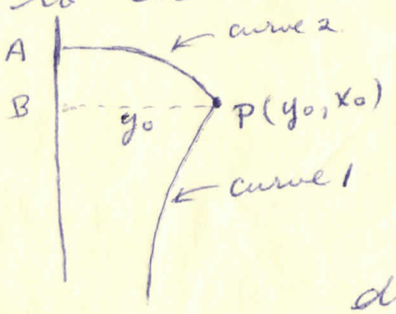
$$= \frac{l}{2}\left(\frac{y_0^2}{l^2-y_0^2}\right) \approx \frac{y_0^2}{2l} \qquad \text{where } l = OA.$$

Now this is "not-quite" correct - in fact it slightly underestimates the vertical drop of the upper plate since the curving of the plate will produce a greater drop than is obtained by this "straight OP" approximation. This does however indicate the way in which the approximate result will develop.

For a closer approximation we might calculate the length of OP from the standard approach

$$OP = \int_0 ds = \int_0^l \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

This leads to some rather formidable integration however - and it proves to be much easier to calculate AB as follows



We define the curve AP as the curve traced out by the point P as the deflection of the side is increased. Our drawing is grossly exaggerated but since the main thrust on P will be ⊥ to the flexible side at P - AP will always be orthogonal to OP at P - ie AP and OP will always intersect at right angles at P.

$$\text{Now} \quad AB = \int_A^B dx = \int_A^P \left(\frac{dx}{dy}\right) dy \quad \text{over curve AP}$$

$$= \int_A^P \left(-\frac{dy}{dx}\right) dy$$

where the integration is still performed along curve AP but we now use the - reciprocal of the slope of curve OP at the same point P.

integration performed along AP

slope of curve OP taken at ~~point~~ intersection point

From - Mechanics and Properties of Matter by Stephenson page 249 - we obtain the result that the deflection y at any point x along the side is given by

$$y = -\frac{W}{6EI_A}\left(3lx^2 - x^3\right)$$

where E ~ Youngs Modulus
W ~ thrust to right causing bending of plate
I_A ~ second moment of area

Let $W/EI_A = k$.

Then $$y_0 = -\frac{k}{6}\left(3l^3 - l^3\right) = -\frac{kl^3}{3}$$

Now $$\left.\frac{dy}{dx}\right|_{x=l} = -\frac{k}{6}\left(6lx - 3x^2\right)\Big|_{x=l} = -\frac{kl^2}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{kl^2}{2} = -\frac{kl^3}{l\,3}\frac{3}{2} = \frac{3y_0}{2l}$$ *Note k is not in this expression which is fortunate as k is a function of W & ∴ related to y.*

$$\therefore AB = \int_0^{y_0}\left(-\frac{3y'}{2l}dy'\right) = -\frac{3}{2l}\frac{y_0^2}{2} = \frac{3}{4}\frac{y_0^2}{l} \quad \text{(equ 1)}$$

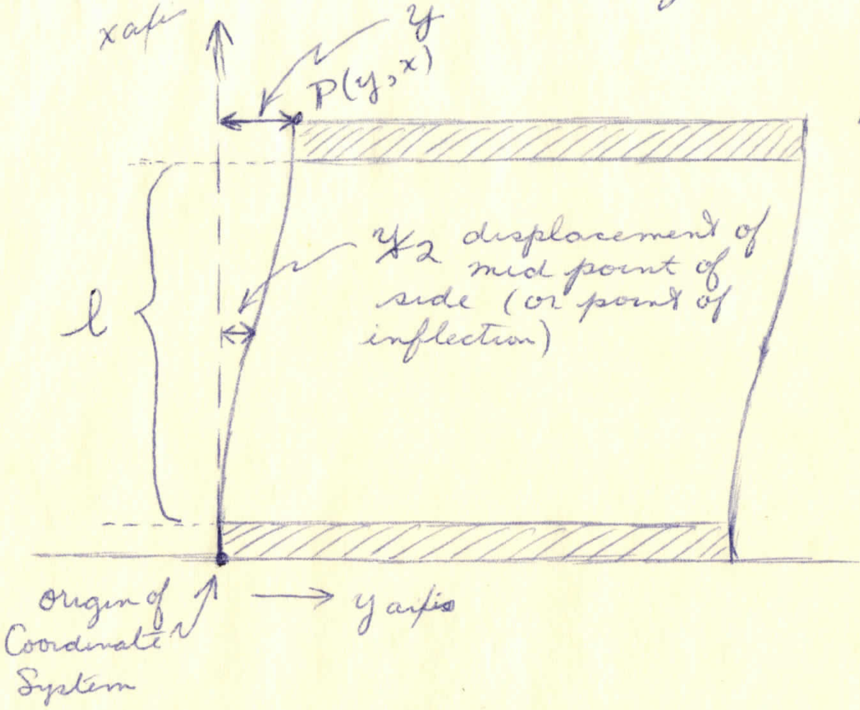
Thus we find that our new approximation yields a result about 50% higher than that obtained by the crude first approximation. It certainly is in the correct direction and seems quite reasonable.

Also - it is of the same form ~ y_0^2

We now set up the expressions for the kinetic energy T and the potential energy V - and we ~~let~~ use the terminology given in the following diagram.

Note - the basic parameter, in terms of which the situation will be described will be the displacement

y of the mass in the horizontal direction



M = mass of upper plate plus any material sitting on it

If we label the coordinates of P as (y, x) then the coordinates of the center of mass are given by $y + Y$, and $x + X$ where Y and X are respectively the distances along the y & x axes from the point P to the Center of Mass. Thus, if the center of mass does not shift during the motion $\dot{X} = \dot{Y} = 0$

and $T = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2)$ (equ-2)

<u>Note</u> - the Kinetic energy is independent of the location of the C of M provided the C of M is not shifting or undergoing jitter - <u>This is a very important qualification on the meaning of the ~~phrase~~ expression - "independent of mass location"</u>. No device is mass location independent if the C of M is shifting & thus "getting into the Kinetic Energy expression above.

The potential energy expression or term is made up of two parts, one the energy in the sides, the other the energy of the mass owing to its vertical position. Now from Stephenson, we find (equ 7.15) that the force required to bend a beam is given by

$$W = \frac{3yEI_A}{l^3}$$

so the potential energy in such a strained system is

$$V = \int_0^y W\,dy' = \frac{3EI_A}{l^3}\frac{y^2}{2} \qquad \text{--- (equ 3)}$$

Applying <u>this expression</u> to <u>our system</u> we must remember that the sides ~~change~~ change curvature at the mid point hence each side must be considered as two "beams" end to end. Thus we must replace l by $l/2$ and y by $y/2$

or

$$V = 4\left(\frac{3EI_A}{(l/2)^3}\left(\frac{y}{2}\right)^2\frac{1}{2}\right) = \frac{12EI_A}{l^3}y^2 \quad \text{-- (equ 4)}$$

4 enters because of 4 "beams" - 2 on each side - each one of which has potential energy stored in an amount $\frac{3EI_A}{\left(\frac{l}{2}\right)^3}\left(\frac{y}{2}\right)^2\frac{1}{2}$

Now for the portion of the potential energy arising from the vertical displacement, if we let the potential energy when $y=0$ be represented by V_0 we have

$$V = V_0 + Mg(l-x) = V_0 - M(AB)$$

distance on first and second diagrams.

$$\cancel{= V_0 + Mgl - Mg}$$

$$V = V_0 - \frac{3Mgy^2}{4l} \quad \text{-- equ 5}$$

Note - y occurs squared - x occurs only in the first power.

We should also note that the last term on the previous page is correct — for treating it in detail we have

$$AB = 2\left(\frac{3}{4}\right)\frac{(y_0/2)^2}{l/2} = \frac{3}{4}\frac{y_0^2}{l}$$

two beams of opposite curvature

Collecting these terms for V we have

$$V = \left(\frac{12EI_A}{l^3} - \frac{3Mg}{4l}\right)y^2 \text{ ------- equ 6}$$

where y is the displacement of P.

Now the terms in T are not quite ready to use (equ 2) since $\dot{x}$ & $\dot{y}$ are not independent. From equ 1

$$AB = l - x = \frac{3}{4}\frac{y^2}{l}$$

$$\text{or } -\dot{x} = \frac{3}{2}y\dot{y}/l$$

Hence from this, and equation 2

$$T = \frac{M}{2}\left\{1 + \left(\frac{3y}{2l}\right)^2\right\}\dot{y}^2 \text{ ------- equ 7}$$

Applying the machinery for Lagranges approach

$$\mathcal{L} = T - V = \frac{M}{2}\left\{1+\left(\frac{3y}{2l}\right)^2\right\}\dot{y}^2 - \left(\frac{12EI_A}{l^3} - \frac{3Mg}{4l}\right)y^2$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = \frac{d}{dt}\left(M\left(1+\left(\frac{3y}{2l}\right)^2\right)\dot{y}\right)$$

$$= M\left(1+\left(\frac{3y}{2l}\right)^2\right)\ddot{y} + \frac{9}{2}My\left(\frac{\dot{y}}{l}\right)^2$$

$$-\frac{\partial\mathcal{L}}{\partial y} = -M\left\{\frac{9}{4}\left(\frac{\dot{y}}{l}\right)^2 y\right\} + \left\{\frac{24EI_A}{l^3} - \frac{3Mg}{2l}\right\}y$$

Thus

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) - \frac{\partial\mathcal{L}}{\partial y} = \boxed{0 = M\left(1+\left(\frac{3y}{2l}\right)^2\right)\ddot{y} + \frac{9M}{4}\left(\frac{\dot{y}}{l}\right)^2 y + \left\{\frac{24EI_A}{l^3} - \frac{3Mg}{2l}\right\}y}$$

equ 8

To state the case bluntly — this equation, in its general form, is not easily solved!

However ~ the situation is not as hopeless as it seems to be at first sight, since approximations can be made which are quite valid and useful.

<u>First</u> - we examine the coefficient of $M\ddot{y}$.

Now if the value of $l = 4''$ and $y_{max} = \frac{1}{4}''$ then $1 + \left(\frac{3y}{2l}\right)^2 \leq 1 + \left(\frac{3}{2 \times 4 \times 4}\right)^2 = 1 + \frac{9}{1024} \simeq 1.009$

Thus the mass ~~M~~ is replaced by an effective mass such that $M < M_{eff} < M(1.009)$. Even though there are no obvious guidelines concerning what ~~the~~ averaging process should be used it is apparent that using the mid value of $M_{eff} = 1.0045\,M$ is not going to introduce a serious error.

<u>Second</u> - we examine the term $\frac{9M}{4}\left(\frac{\dot{y}}{l}\right)^2 y$ relative to the other terms in y. ~~Some~~ simplification is necessary in the basic equation!

Write $1 + \left(\frac{3y}{2l}\right)^2 = H$

" $\frac{24 E I_A}{M l^3} = J$

Divide equation 8 by MH and rewrite as

$$\ddot{y} + \left[J - \frac{3g}{2l} + \left(\frac{3\dot{y}}{2l}\right)^2 \right] \frac{y}{H} = 0 \quad \text{---- eqn 9}$$

[Note - it is apparent (with the equation in this form) that we could introduce a drag term; like the usual form, it would be $\frac{C}{MH}\dot{y}$]

I understand that some actual data could easily be $f = 5$ cycles/sec, $y_{max} = \frac{1}{4}$ inch (or say 0.5 cm). Then for $y = 0.5 \sin \omega t$ we would have the following relationships

$$\omega^2 = (2\pi f)^2 = (10\pi)^2 \approx 1000$$
$$= \left(J - \frac{3g}{2l} + \left(\frac{3\dot{y}}{2l}\right)^2\right)\frac{1}{H}$$

Now $\frac{3g}{2l} \approx \frac{3}{2}\frac{(980)}{10} \approx 150$

$$\left(\frac{3\dot{y}}{2l}\right)^2 \leqslant \left[\frac{3\,\omega\,(0.5)}{2(10)}\right]^2 = \frac{9}{4} \times \frac{1000}{400} \approx 6$$

$$\therefore \quad J \simeq 1000 + 150 - 6 \approx 1150$$

Thus for the case considered here, the $(\dot{y})^2$ term is small at all times, relative to the other terms.

We conclude, for a first approximation, that provided we stick to the working regions considered here, the net result will be that the system undergoes an oscillatory motion which is very close to sinusoidal and which has a frequency shifted slightly from $f = \frac{1}{2\pi}\sqrt{\frac{24EI_A}{Ml^3} - \frac{3g}{2l}}$

Now to get a good approximation to this altered motion we proceed as follows. Assume a solution $y = y_0 \sin\omega t$ and insert this in the equation for the conservation of energy.

ie $T + V = \text{Constant}$. The conservation equation leads to

$$\frac{MH\dot{y}^2}{2} + \left(\frac{12EI_A}{l^3} - \frac{3Mg}{4l}\right)y^2 = \left(\frac{12EI_A}{l^3} - \frac{3Mg}{4l}\right)y_0^2$$

---- equ 10

where the constant is evaluated by use of the condition that $y = y_0$ when $\dot{y} = 0$ (ie at maximum displacement)

Inserting $y = y_0 \sin \omega t$ and $\dot{y} = \omega y_0 \cos \omega t$ into equation 10 we obtain

$$\frac{M}{2}\left(1 + \frac{9 y_0^2}{4 l^2} \sin^2 \omega t\right) y_0^2 \omega^2 \cos^2 \omega t =$$

$$\left(\frac{12 E I_A}{l^3} - \frac{3 M g}{4 l}\right)\Big(\underbrace{y_0^2 - y_0^2 \sin^2 \omega t}_{y_0^2 \cos^2 \omega t}\Big)$$

or $$\frac{M}{2}\left(1 + \frac{9 y_0^2}{4 l^2} \sin^2 \omega t\right) \omega^2 = \left(12 \frac{E I_A}{l^3} - \frac{3 M g}{4 l}\right)$$

Now this shows that the concept of a solution $y = y_0 \sin \omega t$ must be modified to conceive of an ω which itself is a function of time. (the only way in which the above equation is satisfied) This of course is not observable in any instantaneous sense - what we require is an average value of ω. Averaging both sides of the above equation over one period, we obtain (since $\frac{1}{T}\int_0^T \sin^2 \omega t \, dt = \frac{1}{2}$)

$$\frac{M}{2}\left(1 + \frac{9 y_0^2}{8 l^2}\right)(\bar{\omega})^2 = 12 \frac{E I_A}{l^3} - \frac{3 M g}{4 l}$$

or

$$\bar{\omega} = \sqrt{\frac{\frac{24 E I_A}{M l^3} - \frac{3 g}{2 l} - \left(\frac{C}{2M}\right)^2}{1 + \frac{9 y_0^2}{8 l^2}}}$$

A decay term has been added - this is a small term & I have neglected the fact that this term has an H in it. H=1

It appears that the observed ω will be "mass-location independent" and as the oscillation dies down (y_0 gets smaller) the angular frequency ω increases. This is also what one expects on physical grounds - as the interference from the x vibration decays!