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Perturbation Method of Solution

$$M\ddot{x} + R\dot{x} + R'\dot{x}^2 + Kx = 0$$

for $x = A_0$, $\dot{x} = 0$ at $t = 0$

$$\ddot{x} + \frac{R}{M}\dot{x} + \frac{K}{M}x + \frac{R'}{M}\dot{x}^2 = 0$$

$$\ddot{x} + u\dot{x} + vx + w\dot{x}^2 = 0 \qquad (1)$$

Let

$$x = x_0(t) + wx_1(t) + w^2x_2(t) \qquad (2)$$

Substitute 2 into 1, thus

$$\ddot{x}_0 + \underline{w\ddot{x}_1} + w^2\ddot{x}_2 + u\dot{x}_0 + \underline{uw\dot{x}_1} + uw^2\dot{x}_2$$
$$+ vx_0 + \underline{vwx_1} + vw^2x_2 + w(\dot{x} + w\dot{x}_1 + w^2\dot{x}_2)^2 = 0$$

Terms leading to a second-order approximation:

$$\ddot{x}_0 + w\ddot{x}_1 + w^2\ddot{x}_2 + u\dot{x}_0 + uw\dot{x}_1 + uw^2\dot{x}_2 \qquad (3)$$
$$+ vx_0 + vwx_1 + vw^2x_2 + w\dot{x}_0^2 + 2w^2\dot{x}_0\dot{x}_1 = 0$$

The generating solution is found from

w^0: $$\ddot{x}_0 + u\dot{x}_0 + vx_0 = 0$$

Its solution

$$x_0 = A_1'' e^{-\frac{u}{2} + \frac{\sqrt{u^2-4v}}{2}t} + A_2'' e^{-\frac{u}{2} - \frac{\sqrt{u^2-4v}}{2}t} \qquad (4)$$

For

$$\frac{\sqrt{u^2-4v}}{2} = s\sqrt{-1},$$

4 becomes

$$x_0 = e^{-\frac{u}{2}t}\,(p_0 \cos st + q_0 \sin st)$$

(5)

$$\dot{x}_0 = -\frac{u}{2}\,e^{-\frac{u}{2}t}\,(p_0 \cos st + q_0 \sin st)$$
$$\quad + e^{-\frac{u}{2}t}\,(-s\,p_0 \sin st + s\,q_0 \cos st)$$

For the initial condition $x_0 = A_0$, $\dot{x} = 0$ at $t = 0$, we have for the constants p_0, q_0 --

$$p_0 = A_0$$
$$q_0 = \frac{u}{2s} A_0$$

The generating solution 5 becomes

$$x_0 = e^{-\frac{u}{2}t}\,(A_0 \cos st + \frac{u}{2s} A_0 \sin st)$$
$$= A_0\, e^{-\frac{u}{2}t}\,(\cos st + \frac{u}{2s} \sin st)$$

First-order correction terms are found from

w':

$$\ddot{x}_1 + u\dot{x}_1 + v x_1 = -\dot{x}_0^2$$
$$= \cancel{-A_0^2 e^{-ut}(\cos st + \frac{u}{2s} \sin st)^2}$$
$$= -\left[-A_0 e^{-\frac{u}{2}t} \sin st\,(\frac{u^2}{4s} + s)\right]^2$$
$$= -A_0^2 e^{-ut} \sin^2 st\,(\frac{u^2}{4s} + s)^2$$
$$= -A_0^2 (\frac{u^2}{4s} + s)^2 e^{-ut} \sin^2 st \qquad (6)$$

The complementary function of 6 is

$$x_1 = e^{-\frac{u}{2}t}\,(p_1 \cos st + q_1 \sin st)$$

$$u^2 b e^{-ut} + c u^2 e^{-ut}\cos 2st + c u 2s e^{-ut}\sin 2st + c 2 s u e^{-ut}\sin 2st - c 4 s^2 e^{-ut}\cos 2st$$

$$+ d u^2 e^{-ut}\sin 2st - d 2 s u e^{-ut}\cos 2st - d 2 s u e^{-ut}\cos 2st - d 4 s^2 e^{-ut}\sin 2st$$

$$-b u^2 e^{-ut} - c u^2 e^{-ut}\cos 2st - c 2 u s e^{-ut}\sin 2st - d u^2 e^{-ut}\sin 2st + d 2 u s e^{-ut}\cos 2st$$

$$+ va + vb e^{-ut} + vc e^{-ut}\cos 2st + vd e^{-ut}\sin 2st = \frac{F}{2} e^{-ut} - \frac{F}{2} e^{-ut}\cos 2st$$

$u^2 b e^{-ut}$

$u^2 b - u^2$

Equations:

$$va = 0$$

$$vb = \frac{F}{2}$$

$$2suc + (v - 4s^2)d = 0$$

$$(v - 4s^2)c - 2sud = -\frac{F}{2}$$

$$\therefore \quad a = 0 \qquad c = \left(\frac{2su}{v-4s^2}\right)^2 = \frac{-F(v-4s^2)}{2[(v-4s^2)+(2su)^2]}$$

$$b = \frac{F}{2v} \qquad d = \qquad = \frac{F\,2su}{2[(v-4s^2)+(2su)^2]}$$

To find a particular solution of 6 --

Let $F = -A_0^2 \left(\frac{u^2}{4s}+s\right)^2$

$\sin^2 st = \frac{1}{2} - \frac{1}{2}\cos 2st$

Then

$$\ddot{x}_1 + u\dot{x}_1 + v x_1 = Fe^{-ut}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2st\right) \qquad (7)$$

Let

$$x_1 = a + be^{-ut} + ce^{-ut}\cos 2st + de^{-ut}\sin 2st \qquad (7.1)$$

$$\dot{x}_1 = -ube^{-ut} + c(-ue^{-ut}\cos 2st - 2se^{-ut}\sin 2st)$$
$$+ d(-ue^{-ut}\sin 2st + 2se^{-ut}\cos 2st) \qquad (7.2)$$

$$\ddot{x}_1 = u^2be^{-ut} + c(u^2e^{-ut}\cos 2st + u2se^{-ut}\sin 2st$$
$$+ 2sue^{-ut}\sin 2st - 4s^2e^{-ut}\cos 2st) \qquad (7.3)$$
$$+ d(u^2e^{-ut}\sin 2st - 2sue^{-ut}\cos 2st$$
$$- 2sue^{-ut}\cos 2st - 4s^2e^{-ut}\sin 2st)$$

Substituting 7.1, 7.2, 7.3 into 7 and solving for a, b, c, d:

$$a = 0$$

$$b = \frac{F}{2v}$$

$$c = -\frac{F(v-4s^2)}{2[(v-4s^2)+(2su)^2]}$$

$$d = \frac{F(2su)}{2[(v-4s^2)+(2su)^2]}$$

Thus a particular solution of 6 is

$$x_{1p} = \frac{F}{2v} e^{-ut} - \frac{F(v-4s^2)}{2[(v-4s^2)^2+(2su)^2]} e^{-ut} \cos 2st$$

$$+ \frac{F(2su)}{2[(v-4s^2)^2+(2su)^2]} e^{-ut} \sin 2st \qquad (8)$$

when $F = -A_0^2 \left(\frac{u^2}{4s} + s \right)^2$.

The complete solution of 6,

$$x_1 = e^{-\frac{u}{2}t} (P_1 \cos st + Q_1 \sin st) + x_{1p} \qquad (9)$$

For the initial condition $x_1 = 0$, $\dot{x}_1 = 0$ at $t = 0$

$$P_1 = -\left(\frac{F}{2v} + \frac{F(v-4s^2)}{2[(v-4s^2)^2+(2su)^2]} \right)$$

$$(10)$$

$$Q_1 = -\frac{u}{2s} \left(\frac{F}{2v} - \frac{F(v-4s^2)}{2[(v-4s^2)^2+(2su)^2]} \right)$$

$$+ \frac{Fu}{2sv} - \frac{Fu(v-4s^2)}{2s[(v-4s^2)^2+(2su)^2]} - \frac{F(4s^2u)}{2s[(v-4s^2)^2+(2su)^2]}$$

First-order correction is equation 9 with x_{1p} as in equation 8 with F defined there and P_1, Q_1 as in equation 10

Solution to the first-order correction:

$$x(t) = A_0 e^{-\frac{u}{2}t}\left(\cos st + \frac{u}{2s}\sin st\right) + W\left[e^{-\frac{u}{2}t}\left(P_1 \cos st + Q_1 \sin st\right) + x_{1p}\right]$$

where

$$P_1 = -\frac{F}{2v} + \frac{F(v-4s^2)}{2[(v-4s^2)^2+(2su)^2]}$$

$$Q_1 = -\frac{u}{2s}\left(\frac{F}{2v} - \frac{F(v-4s^2)}{2[(v-4s^2)^2+(2su)^2]}\right. + \frac{Fu}{2su} - \frac{Fu(v-4s^2)}{2s[(v-4s^2)^2+(2su)^2]} - \frac{F(4s^2u)}{2s[(v-4s^2)^2+(2su)^2]}$$

$$F = -A_0^2\left(\frac{u^2}{4s}+s\right)^2 = -\frac{A_0^2}{16s^2}(u^2+4s^2)^2$$

$$x_{1p} = \frac{F}{2v}e^{-ut} - \frac{F(v-4s^2)}{2[(v-4s^2)+(2su)^2]}e^{-ut}\cos 2st + \frac{F(2su)}{2[(v-4s^2)^2+(2su)^2]}e^{-ut}\sin 2st$$

$$u = \frac{R}{M}$$

$$v = \frac{k}{M}$$

$$w = \frac{R'}{M}$$

$$\frac{\sqrt{u^2-4v}}{2} = s\sqrt{-1} \qquad \text{when } u^2-4v<0$$

Approx. Solution by Perturbation Method

$$\ddot{x} + u\dot{x} + vx + w\dot{x}^2 = 0, \tag{1}$$

where $u = \frac{R}{M}$, $v = \frac{K}{M}$, $w = \frac{R'}{M}$.

Let (to 2nd order perturbation terms)

$$x(t) = x_0(t) + w x_1(t) + w^2 x_2(t). \tag{2}$$

Substitute 2 in 1:

$$\ddot{x}_0 + w\ddot{x}_1 + w^2\ddot{x}_2 + u\dot{x}_0 + wu\dot{x}_1 + w^2u\dot{x}_2 \tag{3}$$
$$+ vx_0 + wvx_1 + w^2vx_2 + w(\dot{x}_0 + w\dot{x}_1 + w^2\dot{x}_2)^2 = 0$$

Here we use for $(\dot{x}_0 + w\dot{x}_1 + w^2\dot{x}_2)^2$ only the terms $\dot{x}_0^2$, $2\dot{x}_0\dot{x}_1$. Thus collecting coefficients of like powers of w (up to and including the 2nd degree) in 3, we have

$$\ddot{x}_0 + u\dot{x}_0 + vx_0 + (\ddot{x}_1 + u\dot{x}_1 + vx_1 + \dot{x}_0^2)w \tag{4}$$
$$+ (\ddot{x}_2 + u\dot{x}_2 + vx_2 + 2\dot{x}_0\dot{x}_1)w^2 = 0$$

Equate coefficients of powers of w in 4 to 0:

$$w^0: \quad \ddot{x}_0 + u\dot{x}_0 + vx_0 = 0$$
$$w^1: \quad \ddot{x}_1 + u\dot{x}_1 + vx_1 + \dot{x}_0^2 = 0 \tag{5}$$
$$w^2: \quad \ddot{x}_2 + u\dot{x}_2 + vx_2 + 2\dot{x}_0\dot{x}_1 = 0$$

By solving each equation of 5 in succession, we can determine the functions $x_0(t)$, $x_1(t)$, $x_2(t)$ in

$$x(t) = x_0(t) + w x_1(t) + w^2 x_2(t)$$

to second-order perturbation terms.