Mar 15/66

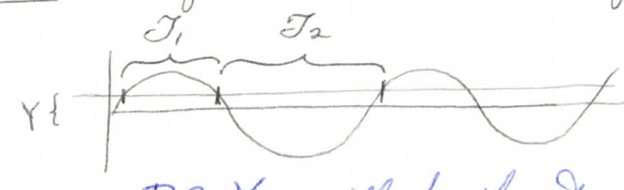
Dear Bill

Here is letter A. It gives an expression for the error in the period of oscillation when you have damping. The damping is assumed to be small. I have calculated it in two approximate ways & they check. Now it took me 6 hours to do this & develop the following questions but I think it was necessary to do so - for it gives you the tool to calculate just what the error amounts to. I DONT KNOW YOUR DATA SO I HAVEN'T TRIED TO MAKE ANY ERROR ESTIMATES. Please let me know what order of error it gives you. Also another reason for pushing on and doing this calculation arises from a worry I have developed -

Question A(1) how do you measure logarithmic decrement in any other than a very sloppy "eyeballing it" method. I have sketched out ~~a more~~ what I believe is an accurate method if you want to worry

Question A(2) about it. Do you want it? - it will take a little more developing but I wont worry about it if you have a good enough procedure. Note In the expression for $\frac{\Delta T}{T}$, m of course refers to the reduced mass of the system or whatever replaces m in the simple expression $\omega = \sqrt{\frac{k}{m}}$.

Question (A3) Do you have a means of reading these time intervals for a variable range of Y where Y is an appreciable fraction of the amplitude?



Sincerely
Ev

P.S. You will shortly get my comments on the papers.

Change of measured period owing to damping

When a damping term $K\frac{dy}{dt}$ is introduced into the equation of S.H.M. there results the well known solution

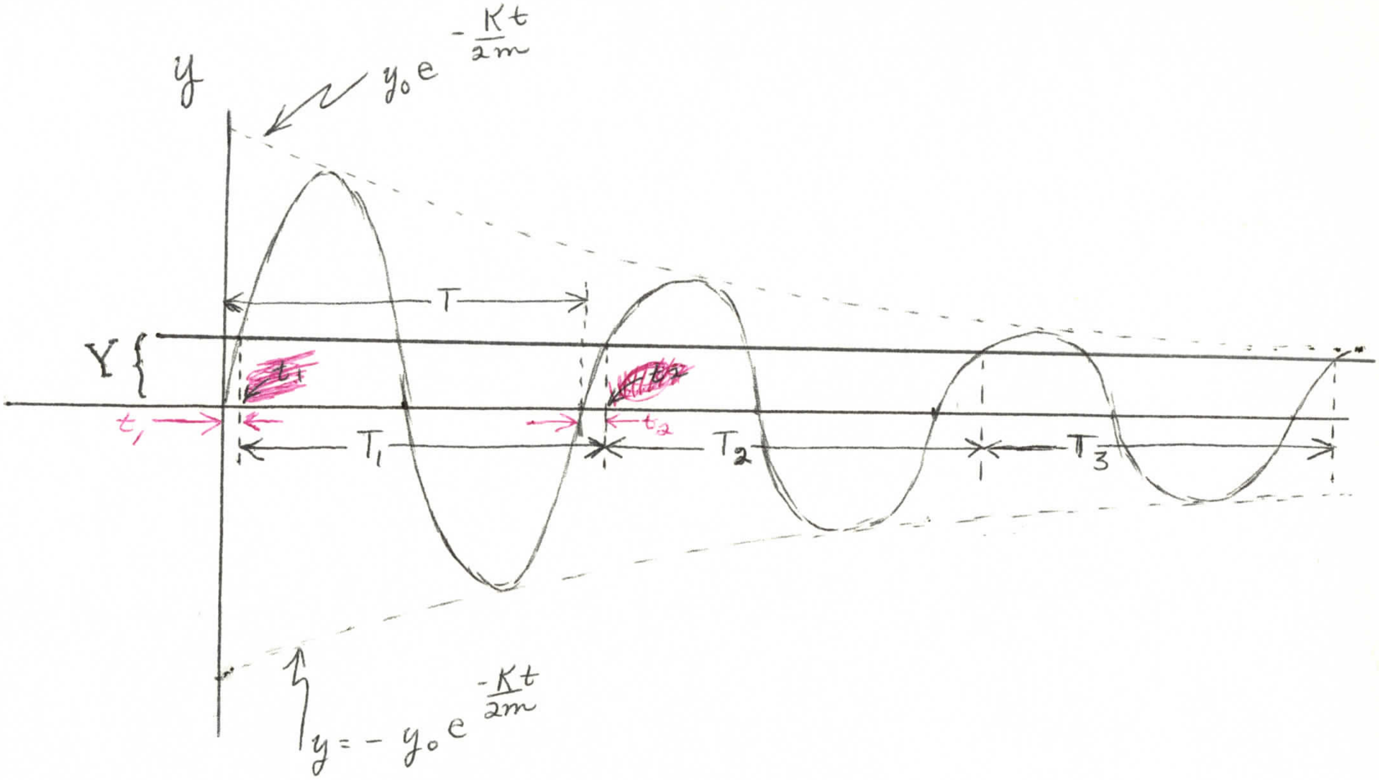
$$y = y_0 e^{-\frac{Kt}{2m}} \sin\left(\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t + \phi\right)$$

where ω_0 is the angular frequency of the un-damped motion and m is the mass of the oscillator. Thus the first apparent effect upon the frequency is that ω_0 is now replaced by $\omega = \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}$ (where we can see $\omega < \omega_0$)

There is however another effect present which depends upon the method of measuring the time when the oscillating mass passes through a given point (which is not the rest position of the mass.) This arises because of the decay of the oscillation i.e. each traversal of this position finds the ~~bob~~ mass in a different position or part of the cycle. Thus the apparent period is not the true period. The diagram on the next page illustrates the this. Let the mass be passing through the rest position at $t=0$, hence $\phi = 0$ and the motion is represented by

$$y = y_0 e^{-\frac{Kt}{2m}} \sin\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t$$

Now suppose that when the displacement is Y and the velocity is positive ($\frac{dy}{dt} > 0$) the timing mechanism is triggered. One can readily observe



from the diagram that $T > T_1$ or T_2 or T_3.

Now it is apparent that if $Y = 0$, $T = T_1 = T_2 = T_3$ hence there would be no error of this type. We may reasonably expect that for this reason (and also because of the obvious inaccuracy of the time measurements performed near the maxima of the curve) that any experimenter is going to attempt to make Y reasonably small. Let us assume that Y is so small that in calculating t_1 we may use the following approximations

$$Kt_1 \lll 1 \quad \text{and} \quad \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t < .15 \text{ radians}$$

Then to a high order of accuracy

$$\cancel{Y = y_0(1 - Kt)}$$

we can convert $Y e^{\frac{Kt_1}{2m}} = y_0 \sin \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t_1$

to $$Y\left(1 + \frac{Kt_1}{2m}\right) = y_0 \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t_1$$

or $$t_1 = \frac{1}{\frac{y_0}{Y}\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2} - \frac{K}{2m}}$$

Let us now proceed to calculate the next point in time at which our displacement is Y and $\frac{dy}{dt} > 0$

This is obviously $t = T + t_2$ hence we have

$$Y = y_0 e^{-\frac{K}{2m}(T+t_2)} \sin \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\,(T+t_2)$$

Now we make the approximation that the logarithmic decrement is small (the damping is not great) and hence $\frac{KT}{2m} << 1$ hence $\frac{Kt_2}{2m} <<< 1$

Also we assume $\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t_2 < 0.15$ radian – so we can safely replace $\sin\theta$ by θ

and we make use of $\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, T = 2\pi$

With all these we ~~obtain~~ rearrange our equation as

$$e^{\frac{KT}{2m} + \frac{Kt_2}{2m}} = \frac{y_0}{Y}\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t_2$$

$$1 + \frac{KT}{2m} = t_2\left(\frac{y_0}{Y}\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2} - \frac{K}{2m}\right)$$

$$t_2 = \frac{1 + \frac{KT}{2m}}{\frac{y_0}{Y}\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2} - \frac{K}{2m}}$$

Now if the error arising from this effect is called ΔT we have $\Delta T = T_1 - T = (T + t_2 - t_1) - T = t_2 - t_1$

$$\therefore \Delta T = \frac{\frac{KT}{2m}}{\frac{y_0}{Y}\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2} - \frac{K}{2m}}$$

$$\Delta T = \frac{T}{\frac{2my_0}{KY}\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2} - 1}$$

Now ΔT is certainly $\ll T$ hence the denominator is certainly much greater than 1 hence

$$\Delta T = \frac{KYT}{2my_0\sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}}$$

$$\boxed{\Delta T = \frac{KY\,T^2}{4\pi m\, y_0}}$$

or $$\boxed{\frac{\Delta T}{T} = \frac{KY}{4\pi m y_0}\,T}$$

Note

$\frac{\Delta T}{T} \propto K$
$\propto Y$
$\propto \frac{1}{y_0}$
$\propto T$

All of which seem reasonable - incidentally the expression checks dimensionally.

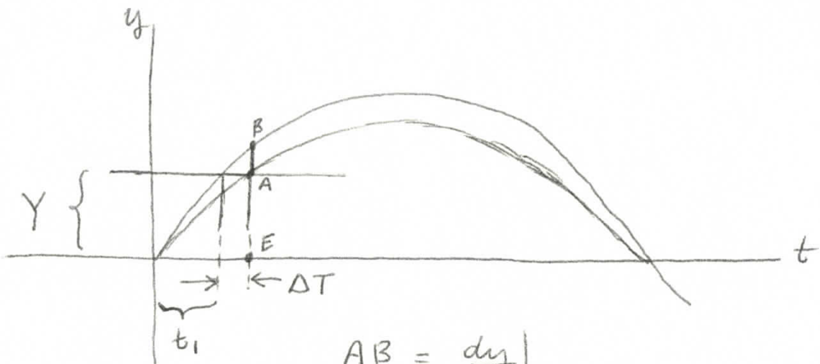
Since K, Y, m & T are properties of the system, one can see that this expression which was calculated for the deviation of the first period from T will hold for later periods if one takes into account the fact that the amplitude of the vibration ~~expression~~ is changing and replaces ~~by replacing~~ y_0 by $y_0 e^{-\frac{Kt}{2m}}$. This need not be a point of worry however since it is assumed that the actual weighing situation is never going to occur in such a manner that the vibration undergoes any considerable damping.

$$\left.\frac{\Delta T}{T}\right|_{\text{at time } t} = \frac{KY}{4\pi m y_0 e^{-\frac{Kt}{2m}}}\,T$$

This shows your fractional error increases with time

CHECK OF ABOVE APPROXIMATION

If one plots two successive ~~ba~~ vibrations, (say the first and second as we have done above) but superimposes them at the point where they cross the ~~y a~~ t axis - he can quickly observe the following relationship



$$\frac{AB}{\Delta T} = \left.\frac{dy}{dt}\right|_{t = t_1 \text{ say.}}$$ same t_1 as before.

$$AB = BE - AE = AE\left(e^{+\frac{KT}{2m}} - 1\right) = \frac{YKT}{2m}$$

since $\frac{BE}{AE} = e^{+\frac{KT}{2m}}$ and $AE = Y$.

Also $$\left.\frac{dy}{dt}\right|_{t=t_1} = y_0 e^{\frac{-Kt_1}{2m}} \left\{ \frac{-K}{2m} \sin \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t_1 + \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2} \cos \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}\, t_1 \right\}$$

Now $\frac{K}{2m} << \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2}$ since damping is small

$\cos\sqrt{}\, t >> \sin\sqrt{}\, t$ $\qquad \cos\sqrt{}\, t \simeq 1$

And $\exp\left(\frac{-Kt_1}{2m}\right) \simeq 1$

$$\frac{dy}{dt} \approx y_0 \sqrt{\omega_0^2 - \left(\frac{K}{2m}\right)^2} = \frac{2\pi y_0}{T}$$

$$\boxed{\therefore \Delta T \approx \frac{YKT}{2m y_0 \frac{2\pi}{T}} = \frac{YKT^2}{4\pi y_0 m}}$$ as before.