PHASE ANGLE. Equation (2.5) for the displacement in oscillatory motion can be written, introducing the frequency relation of Eq. (2.6),

$$x = A \sin \omega_n t + B \cos \omega_n t = C \sin (\omega_n t + \theta) \tag{2.9}$$

where $C = (A^2 + B^2)^{1/2}$ and $\theta = \tan^{-1} (B/A)$. The angle θ is called the *phase angle*.

STATIC DEFLECTION. The static deflection of a simple mass-spring system is the deflection of spring k as a result of the gravity force of the mass, $\delta_{st} = mg/k$. (For example, the system of Fig. 2.4 would be oriented with the mass m vertically above the spring k.) Substituting this relation in Eq. (2.8),

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \tag{2.10}$$

The relation of Eq. (2.10) is shown by the diagonal-dashed line in Fig. 2.5. This relation applies only when the system under consideration is both linear and elastic. For example, rubber springs tend to be nonlinear or exhibit a dynamic stiffness which differs from the static stiffness; hence, Eq. (2.10) is not applicable.

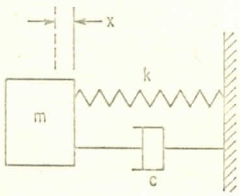

FIG. 2.6. Single degree-of-freedom system with viscous damper.

FREE VIBRATION WITH VISCOUS DAMPING [1,2,3]

Figure 2.6 shows a single degree-of-freedom system with a viscous damper. The differential equation of motion of mass m, corresponding to Eq. (2.4) for the undamped system, is

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.11}$$

The form of the solution of this equation depends upon whether the damping coefficient is equal to, greater than, or less than the *critical damping coefficient* c_c:

$$c_c = 2\sqrt{km} = 2m\omega_n \tag{2.12}$$

The ratio $\zeta = c/c_c$ is defined as the *fraction of critical damping*.

LESS-THAN-CRITICAL-DAMPING. If the damping of the system is less than critical, $\zeta < 1$; then the solution of Eq. (2.11) is

$$x = e^{-ct/2m}(A \sin \omega_d t + B \cos \omega_d t)$$

$$= Ce^{-ct/2m} \sin (\omega_d t + \theta) \tag{2.13}$$

where C and θ are defined with reference to Eq. (2.9). The *damped natural frequency* is related to the undamped natural frequency of Eq. (2.6) by the equation

$$\omega_d = \omega_n(1 - \zeta^2)^{1/2} \quad \text{rad/sec} \tag{2.14}$$

Equation (2.14), relating the damped and undamped natural frequencies, is plotted in Fig. 2.7.

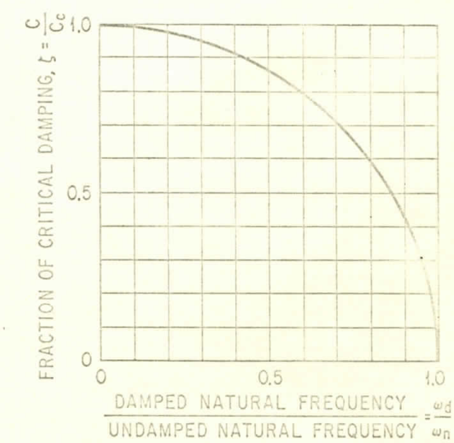

FIG. 2.7. Damped natural frequency as a function of undamped natural frequency and fraction of critical damping.

CRITICAL DAMPING. When $c = c_c$, there is no oscillation and the solution of Eq. (2.11) is

$$x = (A + Bt)e^{-ct/2m} \tag{2.15}$$

GREATER-THAN-CRITICAL-DAMPING. When $\zeta > 1$, the solution of Eq. (2.11) is

$$x = e^{-ct/2m}(Ae^{\omega_n\sqrt{\zeta^2-1}\,t} + Be^{-\omega_n\sqrt{\zeta^2-1}\,t}) \qquad (2.16)$$

This is a nonoscillatory motion; if the system is displaced from its equilibrium position, it tends to return gradually.

LOGARITHMIC DECREMENT. The degree of damping in a system having $\zeta < 1$ may be defined in terms of successive peak values in a record of a free oscillation. Substituting the expression for critical damping from Eq. (2.12), the expression for free vibration of a damped system, Eq. (2.13), becomes

$$x = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \theta) \qquad (2.17)$$

Consider any two maxima (i.e., value of x when $dx/dt = 0$) separated by n cycles of oscillation, as shown in Fig. 2.8. Then the ratio of these maxima is

$$\frac{x_n}{x_0} = e^{-2\pi n\zeta/(1-\zeta^2)^{1/2}} \qquad (2.18)$$

Values of x_n/x_0 are plotted in Fig. 2.9 for several values of n over the range of ζ from 0.001 to 0.10.

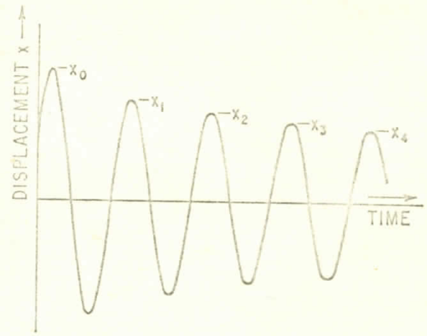

FIG. 2.8. Trace of damped free vibration showing amplitudes of displacement maxima.

The *logarithmic decrement* Δ is the natural logarithm of the ratio of the amplitudes of two successive cycles of the damped free vibration:

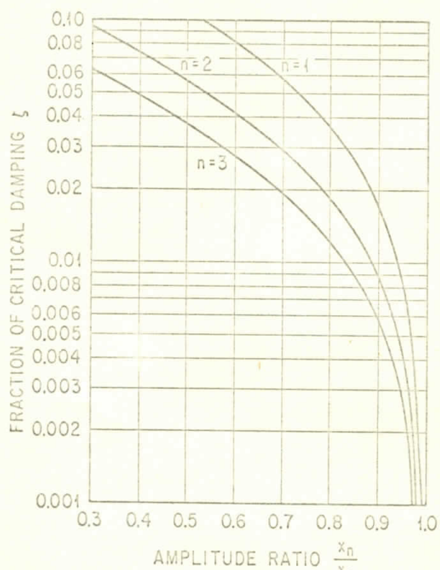

FIG. 2.9. Effect of damping upon the ratio of displacement maxima of a damped free vibration.

$$\Delta = \log\frac{x_1}{x_2} \quad \text{or} \quad \frac{x_2}{x_1} = e^{-\Delta} \qquad (2.19)$$

A comparison of this relation with Eq. (2.18) when $n = 1$ gives the following expression for Δ:

$$\Delta = \frac{2\pi\zeta}{(1-\zeta^2)^{1/2}} \qquad (2.20)$$

The logarithmic decrement can be expressed in terms of the difference of successive amplitudes by writing Eq. (2.19) as follows:

$$\frac{x_1 - x_2}{x_1} = 1 - \frac{x_2}{x_1} = 1 - e^{-\Delta}$$

Writing $e^{-\Delta}$ in terms of its infinite series, the following expression is obtained which gives a good approximation for $\zeta < 0.2$:

$$\frac{x_1 - x_2}{x_1} = \Delta \qquad (2.21)$$

For small values of ζ (less than about 0.10), an approximate relation between