W. E. Baker
16-1917-01
7 June 1966

Equations of Motion for Mass Measuring Device

This device consists of an air-bearing platform resting on a smooth, level surface which is held in a neutral position by an opposing pair of pretensioned coil springs. The platform is displaced from its neutral position and allowed to oscillate. The period of oscillation τ_0 is measured quite accurately, and the mass M inferred from the relation for a simple, undamped harmonic oscillator,

$$M = K\left(\frac{\tau_0}{2\pi}\right)^2 \qquad (1)$$

where K is the total spring rate in force per unit length ~~extension~~ of the spring. In analysis of data obtained with this device at Brooks (AFB), change in period is interpreted as change in mass according to equation (1), even though the change in period may be due to damping effects, coupling with a system having more than one degree of freedom, etc. The apparatus is not arranged so that decay of vibrational amplitude with time can be easily measured. A number of different experiments have been conducted, including oscillation of the air-bearing platform alone, the platform plus a light honeycomb pallet which can support a man or an anthropomorphic dummy, platform plus solid mass in the form of lead weights or shot, platform plus "sails" to increase air damping, and various combinations of the above.

Our purpose here is to write and obtain solutions for the equations of motion of the platform, with either rigid or non-rigid masses attached, accounting for various possible sources of damping or other effects which can alter the fundamental period of oscillation of the platform.

The equation of motion from which eq. (1) is obtained is

$$M\ddot{x} + Kx = 0 \quad \ldots \ldots (2).$$

Motion is simple harmonic with period

$$\tau_0 = 2\pi (M/K)^{1/2}, \quad \ldots \ldots (1a)$$

and amplitude of osillation does not decay with time. The entire platform plus added masses is assumed to be rigid, and dissipative forces (damping) are assumed negligible.. The simplest damping law introduces a force proportional to velocity, and yields the linear equation

$$M\ddot{x} + C_1\dot{x} + Kx = 0 \quad \ldots \quad (3)$$

As for eq. (2), an exact analytic solution to this equation can be obtained*, yielding ~~a decaying~~ vibration which decays exponentially in amplitude, with a period

$$\tau_1 = \frac{\tau_0}{\left(1 - \frac{C_1^2}{4MK}\right)^{1/2}} = \frac{\tau_0}{\left(1 - \frac{C_1^2\tau_0^2}{16\pi^2M^2}\right)^{1/2}} \quad \ldots \quad (4)$$

If we now assume that air damping of the motion is appreciable, but that the platform with attached masses still behaves as a single rigid body, then the equation of motion is

$$M\ddot{x} + C_1\dot{x} + C_2\dot{x}|\dot{x}| + Kx = 0 \quad \ldots \quad (5)$$

The third term in this equation represents air drag forces, and in a steady stream of

* See any standard text on mechanical vibrations for such solutions

air would have the form

$$C_D \cdot S \cdot \tfrac{1}{2}\rho \dot{x}^2 \quad \ldots \ldots \quad (6)$$

$$\text{or} \quad C_2 = \tfrac{1}{2} C_D S \rho \quad \ldots \ldots \quad (6a);$$

The coefficient C_2 is therefore the product of a non-dimensional drag coefficient C_D, an area S normal to direction of motion, and the dynamic pressure $\frac{1}{2}\rho\dot{x}^2$. In eq. (5), the absolute value sign is necessary for the drag force always to oppose the motion, so that it is dissipative over all of each cycle of oscillation.

The drag coefficient C_D is a function of the shape of the object about which the air is flowing, the instaneous velocity $|\dot{x}|$, and other factors such as Reynolds number. For an oscillating body, C_D is often much greater than for a body in steady flow, particularly at low ~~ft~~ peak flow velocities. The values can range as high as $C_D = 12$ for a flat plate, whereas $C_D \approx 2$ is the ~~usual~~ corresponding maximum ~~assumed~~ for steady flow.

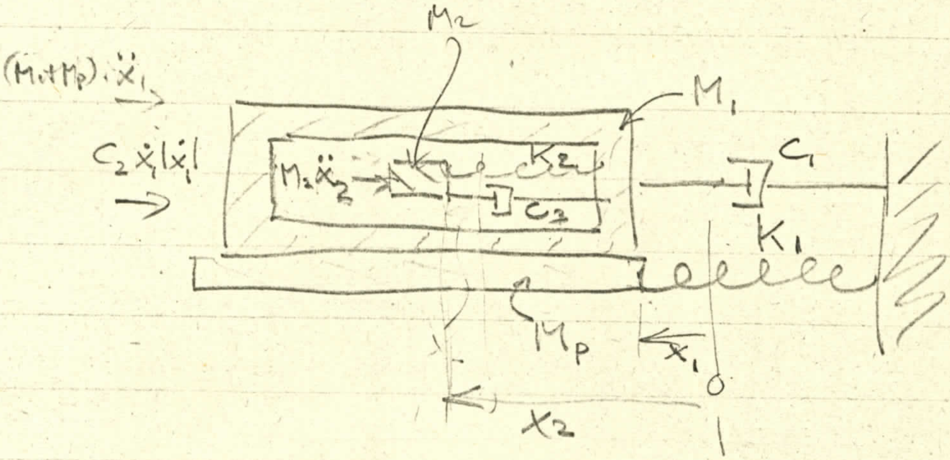
Equation (5) cannot be solved analytically because of the non-linear air damping term. It is, however, quite amenable to digital or analog computer solution. One can expect that change in vibration period will be quite insensitive to change in values of the damping coefficient c_2, so that high accuracy in computation of response is needed. Digital computer solution, by such methods as Runge-Kutta, will therefore probably be preferable to analog computer solution. (If decay of amplitude were measured, response could probably be computed with sufficient accuracy by analog computer.— in fact, an adequate approximate solution for change in amplitude with time has already been obtained by Minorsky*.)

Let us now consider a slightly more complex model to obtain at least an approximate representation of a man on the oscillating platform. We will assume that the man can be represented as a two-mass system, with a relatively small mass M_2 representing his

---

* N. Minorsky, Non-Linear Mechanics, J. W. Edwards, Ann Arbor, Mich., 1947, pp 195 - 196.

viscera being attached to (another mass M_1 representing) the remainder of his body, through a spring and linear damper. Schematically, the man on the oscillating platform (mass M_p) is then represented as shown below.



In the jargon of applied mechanics, this system is a "coupled system with two degrees of freedom". The motion can be represented by two equations which are coupled, as follows:-

$$(M_1+M_p)\ddot{x}_1 + C_1\dot{x}_1 + C_2\dot{x}_1|\dot{x}_1| - C_3(\dot{x}_2-\dot{x}_1) - K_2(x_2-x_1) + K_1x_1 = 0 \qquad (7)$$

$$M_2\ddot{x}_2 + C_3(\dot{x}_2-\dot{x}_1) + K_2(x_2-x_1) = 0 \qquad (8)$$

Again, because of the presence of the non-linear term in eq. (7), this system of equations has no analytic solution. They can probably be solved numerically by digital computer, but at the expense of considerably more effort than for the ~~simpler~~ single equations of motion presented previously. One aspect of considering a system with two degrees of freedom, entirely apart from the effects of dissipative forces, can be illustrated by considering modified forms of eq. (7) and (8) with all damping coefficients set equal to zero. These equations then reduce to

$$M^* \ddot{x}_1 - K_2(x_2 - x_1) + K_1 x_1 = 0 \qquad (7a)$$

$$M_2 \ddot{x}_2 + K_2(x_2 - x_1) = 0 \qquad (8a)$$

where $M^* = M_1 + M_p$. This set of equations can now be solved analytically.* There are two characteristic vibration periods given by

$$\tau = 2 \cdot \sqrt{2}\,\pi \Big/ \left\{ \frac{(K_1+K_2)}{M^*} + \frac{K_2}{M_2} \pm \left[\frac{(K_1+K_2)^2}{M^{*2}} + \frac{2K_2(K_2-K_1)}{M^* M_2} + \frac{K_2^2}{M_2^2} \right]^{1/2} \right\}^{1/2} \qquad (9)$$

---

* See J. P. den Hartog, <u>Mechanical Vibrations</u>, McGraw-Hill Book Co., New York, 1947, pp 103-106.

The longer period, given by the minus sign in eq. (9), is <u>always</u> greater than the period for a rigid man, given by

$$\tau_0 = 2\pi\,(M_T/K_1)^{1/2}, \qquad (1b)$$

where $M_T = M_1 + M_2 + M_p$. As an example, consider the following case:

$M_p = 70/386 = 0.199$ lb sec²/in.

$M_1 = 128/386 = 0.332$ "

$M_2 = 27/386 = 0.0700$ "

$K_1 = 20$ lb/in.

$K_2 = 100$ lb/in

Then, from (9),

2295.7

1428.57

160

$$\tau_1 = 2\cdot\sqrt{2}\,\pi \Big/ \left\{ \frac{120}{0.531} + \frac{100}{0.0700} - \left[\frac{120^2}{0.531^2} + \frac{2\cdot 100\cdot 80}{0.531\cdot 0.0700} + \frac{100^2}{0.0700^2} \right]^{1/2} \right\}^{1/2}$$

$$= 8.8858 \Big/ \{ 225.99 + 1428.57$$

2.5223×10^6

$$- [5.1071\times10^4 + 43.045\times10^4 + 204.08\times10^4]^{1/2} \}^{1/2}$$

$$\tau_1 = 8.8858 \Big/ \{1654.56 - 1588.2\}^{1/2}$$

66.4

$$\tau_1 = \frac{8.8858}{8.1488} = \underline{1.0904} \text{ sec.}$$

0.030050

From (1b), $\tau_0 = 2\pi\,(0.601/20)^{1/2}$

$$\tau_0 = 6.2832 \times 0.17335 = \underline{1.0892} \text{ sec.}$$

If one were to interpret the change in period as an apparent mass increase of a rigid system, ΔM, then this mass increment can be calculated from.

$$\Delta M = \left[\left(\frac{\tau_1}{2\pi}\right)^2 \cdot K_1\right] - M_T \quad \ldots\ldots (10)$$

For our example,

0.030116 $\quad$ 0.60232

$$\Delta M = \left[\left(\frac{1.0907}{6.2832}\right)^2 \cdot 20\right] - 0.60100 = 0.00132$$

$.17354$ $\quad$ lb sec²/in

Or,
$$\frac{\Delta M}{M_T} = \frac{0.00132}{0.601} = \underline{0.00220}$$

Added Mass Effects.

For a body oscillating in any fluid, the period of osillation will be to some extent affected by the presence of the fluid itself, even if one disregards damping effects. Some portion of the fluid will be carried back and forth with the body as it oscillates, yielding a force in phase with acceleration but opposing the motion, so that ~~the~~ apparent mass or inertia is increased. The relationship between added mass and change in period can be easily obtained for a simple rigid oscillator by a modification of eq. (1), which yields

$$\frac{\tau_0}{\tau_A} = \left(\frac{1}{1 + \frac{M_A}{M_0}} \right)^{1/2} \qquad (\ )$$

where τ_A is period with added mass, τ_0 period in vacuum, M_A is mass increment or added mass of fluid, and M_0 mass of the rigid body. The quantity M_A is a function of the density of the fluid, geometry of the osillating body, and to some extent of the amplitude and frequency of oscillation. ~~and a~~

$-3+1-2$

$K_{8cps} = 4(9.9)(64)(.0684)$

$= 173\ LB/in$

$K_v \cong 43.4 - 173\ LB/in$

$43.4 \frac{LB}{in} \quad \frac{454\ gram}{LB} \quad \frac{in}{2.54\ cm}$

$\frac{43.4(.454)}{2.54}$

$f_0 = 4 - 8\ cps$

for Abdominal mass

$2.2\ \frac{LB}{kilo}$

$26.4^{\#}$ viscera

$m_v = \frac{26.4\ LB\ sec^2}{386\ in} = .0684$

$\omega = \sqrt{\frac{K}{m}} \qquad \omega = 2\pi f$

$K = m\omega^2$

$= 4\pi^2 f^2 m$

$+2+1-2$

$= 4(9.9)(16)\frac{1}{sec^2}(.0684)\frac{LB\ sec^2}{in}$

$K_{4cps} = 43.4\ LB/in$